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Fusion potentials: I

P Di Francesco and J-B Zuber

Service de Physique Théorique de Saclay, Laboratoire de la Direction des Sciences de la Matière du Commissariat à l'Energie Atomique, F-91191 Gif sur Yvette Cedex, France

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Abstract. We reconsider the conjecture by Gepner that the fusion ring of a rational conformal field theory is isomorphic to a ring of polynomials in n variables quotiented by an ideal of constraints that derive from a potential. We show that, in a variety of cases, this is indeed true with *one-variable* polynomials.

1. Introduction

The fusion properties constitute an essential piece of information on a rational conformal field theory (RCFT). A few years ago, Verlinde [1] was able to express the fusion rules in terms of the unitary matrix S that encodes the modular transformations of the characters of the RCFT

$$N_{ij}^{k} = \sum_{l} \frac{S_{il} S_{jl} S_{kl}^{*}}{S_{1l}} \tag{1.1}$$

where '1' refers to the identity operator, and the labels $i, \ldots, l$ run over n values corresponding to the primary fields of the (extended) chiral algebra of the RCFT [2]. The fusion coefficients N_{ij}^{k} are the structure constants of a commutative and associative algebra

$$A_i A_j = \sum_{k} N_{ij}{}^{k} A_k. \tag{1.2}$$

The matrices N_i defined by

$$(N_i)_{jk} = N_{ij}^{k} \tag{1.3}$$

themselves form a representation of the fusion algebra

$$N_i N_j = \sum_{k} N_{ij}^{k} N_k \tag{1.4}$$

as follows from the unitarity of the matrix S; this expresses the associativity property of the algebra (1.2). Relation (1.1) implies that the matrix S diagonalizes the matrices N_i and that their eigenvalues are of the form

$$\gamma_i^{(l)} = S_{il}/S_{1l}. \tag{1.5}$$

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The general study of these fusion algebras [3] and their classification have been the object of much work [4]. In particular, the possibility that they may be represented by sets of polynomials has been considered [4]. In fact, a fusion algebra is a very special associative and commutative algebra as it possesses a selected basis corresponding to the primary fields of the theory in which the structure constants are *non-negative integers*. For these reasons, one is unwilling to trade this basis for another (except in those cases where there is symmetry between several primary fields). In addition one wants to stress the addition and internal multiplication of fields rather than their multiplication by scalars, i.e. the structure of *ring* rather than that of *algebra*.

Recently, Gepner [5] conjectured that in any RCFT, the fusion ring is isomorphic to a *ring of polynomials in p variables quotiented by an ideal of constraints that derive from a* potential. He was able to prove this for theories with an SU(N) current algebra, for which it is natural to take the $p = N-1$ variables associated with the fundamental representations. This has since been extended to several other cases [6–9].

Particularly interesting are faithful representations which have a trivial kernel, that is in which no linear combination of the generators is represented by the zero polynomial. In the present paper we discuss the possibility that the fusion ring may be faithfully represented by a polynomial ring in a single variable that corresponds to one of the primary fields of the theory and is subject to a polynomial constraint that may be integrated to a potential. We shall first give a simple necessary and sufficient condition for this to happen, namely that the eigenvalues of one of the fusion matrices be non-degenerate. Curiously, cases such as the models with an SU(3) current algebra that are known to have a fusion ring described by polynomials in *two* variables, turn out to satisfy this condition and may thus be also described by polynomials in a single variable. Even when the previous condition is not fulfilled, in many cases, there is a way out that still enables us to use a single variable, at the expense of defining more carefully what is meant by linear combination in the above definition of a faithful representation. This will be illustrated on the minimal ($N = 0$) models, and on the D series of SU(2) models ($\mathbb{Z}_2$ orbifolds).

2. A necessary and sufficient condition for a one-variable polynomial ring

There exists a class of RCFTs in which the existence of a representation of the fusion algebra by polynomials in a single variable may be ascertained. Assume that among the matrices N_i, $i = 1, \ldots, n$, there exists at least one, call it N_f, with only non-degenerate eigenvalues. In other words, the numbers $\gamma_f^{(l)}$ are all distinct. Any other N_i may be diagonalized in the same basis as N_f and there exists a unique polynomial $P_i(x)$ of degree at most $n-1$ such that its eigenvalues $\gamma_i^{(l)}$ read

$$\gamma_i^{(l)} = P_i(\gamma_f^{(l)}) \tag{2.1}$$

P_i being given by the Lagrange interpolation formula. Therefore, any N_i may be written as

$$N_i = P_i(N_f)$$

with a polynomial P_i; as both N_i and N_f have integral entries, $P_i(x)$ must have rational coefficients.

The $n \times n$ matrix N_f, on the other hand, satisfies its characteristic equation $\mathcal{P}(x) = 0$, that is also its minimal equation, as N_f has no degenerate eigenvalues. The constraint on N_f is thus

$$\mathcal{P}(N_f) = 0 \tag{2.3}$$

that may be integrated to yield a 'potential' $\mathcal{V}(x)$,

$$\mathcal{V}'(x) = \mathcal{P}(x) \tag{2.4}$$

that is a polynomial of degree $n+1$.

This situation is far from exceptional. We illustrate it with a few examples.

(i) Ising model. Take for f the spin σ field. The fields $\mathbf{1}$, σ and ϵ obey fusion rules

$$\sigma \cdot \sigma = \mathbf{1} + \epsilon \qquad \sigma \cdot \epsilon = \sigma \qquad \epsilon \cdot \epsilon = \mathbf{1} \tag{2.5}$$

that may be represented by the polynomials in the variable σ

$$1 \qquad \sigma \qquad \epsilon = \sigma^2 - 1 \tag{2.6}$$

subject to the constraint

$$\sigma^3 = 2\sigma \tag{2.7}$$

that is derived from the potential

$$\mathcal{V}(\sigma) = \tfrac{1}{4}\sigma^4 - \sigma^2.$$

(ii) $\mathrm{SU}(2)_k$. This case is well known. (We are considering here the 'diagonal' case, labelled by the A_{k+1} Dynkin diagram.) At level k, there are $k+1$ representations labelled by the integer $n = 2j$, $0 \leqslant n \leqslant k$ and the fusion rules are represented by the multiplication of Chebishev polynomials of the second kind, $P_n(x)$:

$$P_n(2\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \qquad 0 \leqslant n \leqslant k \tag{2.9}$$

with the constraint that

$$P_{k+1}(x) = 0. \tag{2.10}$$

(iii) $\mathrm{SU}(3)_k$. We take for f one of the two fundamental representations. The eigenvalues are known to be of the form

$$\gamma_f^{(\lambda)} = \exp \mathrm{i}\alpha_1 + \exp \mathrm{i}\alpha_2 + \exp -\mathrm{i}(\alpha_1 + \alpha_2) \tag{2.11}$$

where the angles α_1 and α_2 read

$$\alpha_1 = 2\pi \frac{\lambda_1 - \lambda_2}{3(k+3)} \qquad \text{and} \qquad \alpha_2 = 2\pi \frac{\lambda_1 + 2\lambda_2}{3(k+3)}, \lambda_1, \lambda_2 \geqslant 1 \qquad \lambda_1 + \lambda_2 \leqslant k+2. \tag{2.12}$$

It is a tedious but straightforward exercise of trigonometry to check that two numbers of the form (2.11) cannot be degenerate (see appendix A for the detailed proof). (In contrast, for $\mathrm{SU}(4)_2$, it is easy to see that zero is doubly degenerate if we again choose the fundamental representation for f.) We thus conclude that the fusion ring of SU(3) current algebras may be represented by one-variable polynomials. As it is also naturally represented by two-variable polynomials [5] (the two variables being associated with the two fundamental representations), we have here a first instance of the non-uniqueness of this description of fusion rings by polynomial rings. For the sake of definiteness, we give the expressions for the various fields† of $SU(3)_2$ labelled by their indices $\lambda_1, \lambda_2 \geqslant 1$, $\lambda_1 + \lambda_2 \leqslant 4$

$$\begin{aligned} &\Phi_{(1,1)} = 1 \qquad \Phi_{(2,1)} = x \qquad \Phi_{(3,1)} = \tfrac{1}{2}(5x^2 - x^5) \\ &\Phi_{(1,2)} = \tfrac{1}{2}(x^5 - 3x^2) \qquad \Phi_{(2,2)} = \tfrac{1}{2}(x^3 - 1) \\ &\Phi_{(1,3)} = \tfrac{1}{2}(x^4 - 3x) \end{aligned} \tag{2.13}$$

with the constraint

$$x^6 - 4x^3 - 1 = 0 \tag{2.14}$$

arising from the potential $V = \frac{1}{7}x^7 - x^4 - x$. Note that the $\mathbb{Z}_3$ charge (the 'triality') of each field may be read off the degrees modulo 3 in the monomials of its expression in terms of x.

Although the number of variables and the form of the potential differ from that given by Gepner, the ultimate content of the constraint, namely the absence of certain representations in the Kac–Moody algebra, is the same and the constraint and the polynomial representation of the fields match after elimination of one variable in Gepner's formulation. Keeping the case of $SU(3)_2$ for illustration, we recall that the two-variable potential of Gepner reads

$$V(x, y) = \tfrac{1}{5}x^5 - x^3y + x^2 + xy^2 - y \tag{2.15}$$

from which follow the constraints

$$x^4 - 3x^2y + 2x + y^2 = 0 \tag{2.16a}$$

$$x^3 - 2xy + 1 = 0 \tag{2.16b}$$

that express the absence of representations corresponding to Young tableaux with more than two columns. The fields are represented as

$$\begin{aligned} &\Phi_{(1,1)} = 1 \qquad \Phi_{(2,1)} = x \qquad \Phi_{(3,1)} = x^2 - y \\ &\Phi_{(1,2)} = y \qquad \Phi_{(2,2)} = xy - 1 \\ &\Phi_{(1,3)} = y^2 - x. \end{aligned} \tag{2.17}$$

Eliminating y between the two equations (2.16) we find the equation (2.14) in x. The latter may be written as $x(x^5 - 4x^2) = 1$, implying that x may be inverted on the ring. This makes it possible to solve in y using (2.16b) and one recovers all the expressions (2.13).

Conversely let us suppose that one of the Ns, call it N_f, generates the n matrices $N_i = P_i(N_f)$ as linearly independent polynomials on $\mathbb{C}$. If N_f had some degenerate eigenvalue, then its minimal polynomial $\mathcal{M}$ would be of degree at most $n - 1$, which means that one could construct at most $n - 1$ linearly independent polynomials of N_f, in contradiction with the above. Therefore the condition that some N has non-degenerate

† We shall henceforth make a slight abuse of notation and denote in the same way the fields Φ, regarded as generators of the fusion algebra, and their representatives.

eigenvalues is necessary and sufficient to ensure that the fusion ring is faithfully represented by one variable polynomial.

One could object that even if all Ns have degenerate eigenvalues, one could still take some linear combinations thereof having all eigenvalues distinct. This possibility will be discarded in the present study in which we insist that the variable of the polynomials corresponds to one of the primary fields of the theory; in other words, the polynomial $P_1(x) = x$ is a representative of one of the fields.

We have stressed in the introduction the importance of the selected basis of the fusion ring. This explains why situations described by potentials of the same degree, for instance $SU(3)_2$ of equation (2.13), with potential (2.14) $V(x) = \frac{1}{7}x^7 - x^4 - x$, and $SU(2)_5$ with potential (2.10) $W(x) = \frac{1}{7}x^7 - x^5 + 2x^3 - x$, and thus 'local rings' of the same dimension are regarded as inequivalent. In fact, the potential is not a sufficient information to reconstruct the fusion ring with its special basis and its integral structure constants.

Let us now examine what happens when the necessary and sufficient condition is not satisfied. As already mentioned, one cannot then represent all fields by one-variable polynomials linearly independent over $\mathbb{C}$. It turns out that in many cases, one may represent them by polynomials with coefficients in $\bar{\mathbb{Q}}$, some extension of $\mathbb{Q}$ by some algebraic number(s), the polynomials being linearly independent over $\mathbb{Q}$. This seems sensible since the operations in the fusion ring imply only *integral* combinations of the fields. We shall now present two families of RCFTs for which this happens: the minimal models and the non-diagonal 'D' SU(2) models.

3. Minimal models

In the generic (p, q) minimal model (that is with a central charge $c = 1 - 6(p-q)^2/pq$), the condition of the previous section is not satisfied and the fusion matrices N_i have degenerate eigenvalues. We shall now see that fusion rules can nevertheless be faithfully represented by one-variable polynomials with coefficients in some algebraic extension of $\mathbb{Q}$.

Let us consider the most general minimal theory $(2p+1, q)$ (one of the integers has to be odd as they are coprimes). Its primary fields $\Phi_{(r,s)}$ are labelled by integer Kac indices $1 \leqslant r \leqslant 2p$, $1 \leqslant s \leqslant q-1$, and satisfy the reflection symmetry property: $\Phi_{(2p+1-r,q-s)} = \Phi_{(r,s)}$, leaving only $p(q-1)$ independent primaries. One can generate the fusion of these fields in a very simple way:

(i) impose, respectively, the $SU(2)_{2p-1}$ and $SU(2)_{q-2}$ fusion rules on the fields $\Phi_{(r,1)}$, $1 \leqslant r \leqslant 2p$, and $\Phi_{(1,s)}$, $1 \leqslant s \leqslant q-1$;

(ii) deduce the fusions of the fields $\Phi_{(r,s)} = \Phi_{(r,1)} \times \Phi_{(1,s)}$ by imposing the reflection symmetry property.

From this procedure it is clear that all the fields in the Kac table can be realized as polynomials of the two basic fields $A \equiv \Phi_{(1,2)}$ and $B \equiv \Phi_{(2,1)}$ provided the above reflection symmetry is satisfied. More precisely, if we use the Chebishev polynomials P_n of the second kind defined in (2.9)

$$P_n(2\cos x) = \sin(n+1)x/\sin x$$

then we have

$$\begin{aligned} \Phi_{(1,s)} &= P_{s-1}(A) \\ \Phi_{(r,1)} &= P_{r-1}(B) \qquad (3.1) \\ \Phi_{(r,s)} &= P_{s-1}(A)P_{r-1}(B) \end{aligned}$$

and the minimal equations for A and B read

$$P_{q-1}(A) = P_{2p}(B) = 0. \tag{3.2}$$

To get the correct fusion rules, we have to impose the reflection symmetry property

$$P_{s-1}(A)P_{r-1}(B) = P_{q-1-s}(A)P_{2p-r}(B). \tag{3.3}$$

In particular, for $s = q - 1$ and $r = 1$, we must have

$$P_{q-2}(A) = P_{2p-1}(B). \tag{3.4}$$

It is possible to prove that (3.2), (3.4) are sufficient to ensure the reflection symmetry (3.3). At this stage, we have obtained a perfectly satisfactory representation of the fusion ring by polynomials in two variables subject to the constraints (3.2)–(3.4).

In a search for representations by one-variable polynomials, we look for a solution of (3.4) in the form

$$B = \alpha P_{q-2}(A) \tag{3.5}$$

where α is a constant to be determined. Thanks to the $\mathrm{SU}(2)_{q-2}$ fusion rules, we have $P_{q-2}(A)^2 = P_0(A) = 1$, therefore our ansatz for B satisfies $B^2 = \alpha^2 P_0(A) = \alpha^2$. The Chebishev polynomials of even (respectively odd) order are even (respectively odd), therefore the minimal equation for B implies $P_{2p}(\alpha) = 0$. Multiplying both sides of (3.4) by B, and using the $\mathrm{SU}(2)_{q-2}$ and $\mathrm{SU}(2)_{2p-1}$ fusion rules, we get

$$BP_{2p-1}(B) = P_{2p-2}(B) = P_{2p-2}(\alpha) = \alpha P_{q-2}(A)^2 = \alpha \tag{3.6}$$

so that $P_{2p-2}(\alpha) = P_1(\alpha)$. Recursively, we get $P_{2p-1-m}(\alpha) = P_m(\alpha)$, $m = 0, 1, \ldots, p$, which can all be obtained from the single equation

$$(P_p - P_{p-1})(\alpha) = 0. \tag{3.7}$$

Pick any solution of (3.7), then the reflection symmetry property is ensured. Namely for r odd

$$\begin{aligned}
&P_{2p-1-r}(B) = P_{2p-1-r}(\alpha) = P_r(\alpha)\\
&P_r(B) = \frac{P_r(\alpha)}{\alpha}B = P_r(\alpha)P_{q-2}(A)\\
&P_{2p-1-r}(B)P_{q-2-s}(A) = P_r(\alpha)P_{q-2-s}(A) = P_r(\alpha)P_{q-2}(A)P_s(A) = P_r(B)P_s(A)
\end{aligned} \tag{3.8}$$

and for r even

$$\begin{aligned}
&P_{2p-1-r}(B) = [(P_{2p-1-r}(\alpha))/\alpha]B = P_r(\alpha)P_{q-2}(A)\\
&P_r(B) = P_r(\alpha)\\
&P_{2p-1-r}(B)P_{q-2-s}(A) = P_r(\alpha)P_{q-2}(A)P_{q-2-s}(A) = P_r(\alpha)P_s(A) = P_r(B)P_s(A).
\end{aligned} \tag{3.9}$$

We also have to choose the solution α of (3.7) in order for the fields to be linearly independent over $\mathbb{Q}$. Using the defining relation (2.9), (3.7) is easily solved, and we choose

$$\alpha = 2\cos\left(\frac{\pi}{2p+1}\right). \tag{3.10}$$

The Kac table can be described in the following way: the first column $\Phi_{(1,s)}$ consists of $(1, P_1(A), P_2(A), \ldots, P_{q-2}(A))$ where all fields are independent; the second column of $\alpha(P_{q-2}(A), P_{q-3}(A), \ldots, 1)$, the third one of $P_2(\alpha)(1, P_1(A), \ldots, P_{q-2}(A))$, etc. Hence the linear independence over $\mathbb{Q}$ of $(1, P_1(\alpha), P_2(\alpha), \ldots, P_{p-1}(\alpha))$ automatically implies the linear independence of all the fields in the first p columns of the Kac table, i.e. of all the primaries in the theory.

We have thus completed the proof that the fusion rules of the minimal models may be represented faithfully on a ring of polynomials in the variable A. We may illustrate it with the case of the $(4, 5)$ minimal model (the tricritical Ising model). There are six fields that may be represented according to the previous method (Beware! here the roles of r and s have to be interchanged in the previous formulae)

$$\begin{aligned} &1 = \Phi_{(1,1)} = 1 \qquad \Phi_{(1,2)} = \alpha(A^2 - 1) \\ &\Phi_{(2,1)} = A \qquad \Phi_{(2,2)} = \alpha A \\ &\Phi_{(3,1)} = P_2(A) = A^2 - 1 \qquad \Phi_{(3,2)} = \alpha \end{aligned} \tag{3.11}$$

where α is a primitive root of $P_2(\alpha) - P_1(\alpha) = \alpha^2 - \alpha - 1 = 0$, i.e. $\alpha = 2\cos l\pi/5$, $l = 1$ or 3 and A satisfies the constraint

$$P_3(A) = A^3 - 2A = 0. \tag{3.12}$$

In fact there is another representation of the $(4, 5)$ fusion ring where the variable corresponds to the $(1, 2)$ field

$$\begin{aligned} &1 = \Phi_{(1,1)} = 1 \qquad \Phi_{(1,2)} = B \\ &\Phi_{(2,1)} = \frac{1}{\sqrt{2}}(B-1)(B^2+B-1) \qquad \Phi_{(2,2)} = \frac{1}{\sqrt{2}}(B^2+B-1) \\ &\Phi_{(3,1)} = P_3(B) = B(B^2-2) \qquad \Phi_{(3,2)} = P_2(B) = B^2 - 1 \end{aligned} \tag{3.13}$$

subject to the constraint

$$P_4(B) = B^4 - 3B^2 + 1 = 0. \tag{3.14}$$

The choice of the determination of the square root of 2 is arbitrary and its change reflects the fact that the fusion algebra admits a $\mathbb{Z}_2$ automorphism under which $\Phi_{(2,s)}$, $s = 1, 2$ are odd.

This solution is not unique: there is another way of representing this fusion algebra by polynomials of B, with the same expression for the $\mathbb{Z}_2$ even fields, but

$$\Phi_{(2,1)} = \frac{1}{\sqrt{10}}(B-3)(B^2+B-1) \qquad \Phi_{(2,2)} = \frac{1}{\sqrt{10}}(1-2B)(B^2+B-1). \tag{3.13$'$}$$

As further evidence of this non-uniqueness, we shall now establish that one may also represent the fusion ring by polynomials in a variable corresponding to the $\phi = \Phi_{(2,2)}$ field, the 'order parameter'. When the two integers labelling the minimal model are odd, this follows from the analysis carried out in section 2, see the end of the footnote on p 1448. So let us concentrate on the case where the integer q is even, in particular the unitary models where $2p + 1$ and q have to be consecutive integers. Then, it is well known (section 8.5 of

[10]) that the fusion algebra admits a $\mathbb{Z}_2$ automorphism (this is what we have just used in the tricritical Ising model) under which the fields $\Phi_{(r,s)}$, s even, are odd. These $N = p(\frac{1}{2}q - 1)$ fields thus form a subset of the fusion ring. Call $\phi = \Phi_{(2,2)}$ and consider all its odd powers ϕ^{2n-1}, $1 \leqslant n \leqslant N$, they are $\mathbb{Z}_2$ odd, thus linear combinations of the N fields $\Phi_{(r,s)}$, s even. This linear system may be inverted† to yield polynomial expressions (in $\mathbb{Q}[\phi]$) of all these Φ. In particular $\Phi_{(1,2)} = P(\phi)$, but we have seen above that all the fields may be expressed polynomially in terms of $A = \Phi_{(1,2)}$.

For example, in the (3, 4) (Ising) model, the constraint on ϕ is of degree 3, integrating to a quartic potential, for the (4, 5) (tricritical Ising), the ϕ-potential is sextic. Up to this point, these degrees seem to agree with those expected for the Landau–Ginsburg potentials by the argument of Zamolodchikov [11]. Unfortunately, this breaks down for the higher minimal models. In general for the $(m, m+1)$ unitary minimal model we expect a degree $2N + 2 = [(m^2 + 1)/2] - m + 2$ whereas Zamolodchikov's potential is of degree $2m - 2$.

Note that all the irrational coefficients that have appeared, α in (3.10), or $\sqrt{2}$ or $\sqrt{10}$ in (3.13), (3.13′) belong to the field $\mathcal{U}$ obtained by adjoining to $\mathbb{Q}$ all the eigenvalues $\gamma_i^{(\alpha)}$ of all the Ns. This is not an accident, and we shall now prove this fact in general. Suppose we have found a representation of the fusion algebra of an RCFT on a ring of polynomials $P_i(X)$ quotiented by the constraint $V'(X) = 0$. As before, we assume that one of the Ps, call it P_f, is simply $P_f(X) = X$, and that the constraint is the minimal polynomial of degree p satisfied by N_f: in other words, the roots of V' are all distinct, $x_0^{(l)}$. Now the $P_i(x_0^{(l)})$ form a one-dimensional representation of the fusion algebra, thus are nothing other than the eigenvalues $\gamma_i^{(l)}$. In particular $x_0^{(l)} = \gamma_f^{(l)}$. If we regard this set of equations as a linear system in the coefficients of the polynomial P_i

$$P_i(x_0^{(l)}) = \sum_{k=0}^{p-1} P_{ik} x_0^{(l)k} = \gamma_i^{(l)}$$

this system may be inverted, since its determinant is the Vandermonde determinant of the distinct roots $x_0^{(l)}$, and gives for the coefficients P_{ik} values in the field $\mathcal{U}$ generated by the $\gamma_i^{(l)}$.

It would seem possible to reduce further the one-variable polynomial representation by substituting for the variable X one of the roots $x_0^{(l)}$ of $V'(X)$. This would lead to one-dimensional representations of the fusion ring. If the polynomial V' of degree p is reducible on $\mathbb{Q}$, however, the choice of a root $x_0^{(l)}$ amounts to setting to zero a factor of V' of degree less than p, resulting in an unfaithful representation. In all the cases relative to minimal models considered above, the constraint is indeed a reducible polynomial on $\mathbb{Q}$: $P_{2k+1}(X)$ factors out X whereas

$$P_{2k}(X) = (-1)^k (P_k(X) - P_{k-1}(X))(P_k(-X) - P_{k-1}(-X)).$$

† That this linear system may be inverted can be seen by proving that among the eigenvalues of $\phi = \Phi_{(2,2)}$, of the form $\gamma^{(r,s)} = 4\cos\pi r/(2p+1)\cos\pi s/q$, only zero is degenerate (p times); it follows that the minimal polynomial in ϕ is of degree $p(q-1) - (p-1) = 2N + 1$, whereas the linear dependence of $\phi, \ldots, \phi^{2N-1}$ would lead to a smaller degree. Degeneracies other than zero cannot occur for number theoretic reasons: if $\gamma^{(r,s)} = \gamma^{(r',s')}$, the ratio $\cos(\pi r/(2p+1))/\cos(\pi r'/(2p+1))$ has to be a rational, which can occur only in the trivial cases $r = r'$ or $r' = 2p+1-r$. We relegate the proof of these assertions to appendix B. The same argument shows that for non-unitary (p, q) minimal models with both p and q odd, or for the $(3, q)$ models, $\Phi_{(2,2)}$ has non-degenerate eigenvalues and is thus a generator of the fusion algebra.

4. The SU(2) models of D-type

For even level $k = 2n$, the $SU(2)_k$ fusion admits a $\mathbb{Z}_2$ automorphism with the fields of half-integer spin ($2j$ odd) odd; the orbifold with respect to this symmetry gives rise to the so-called D_{n+2} models [12, 13]. For $k = 0 \bmod 4$, i.e. $n = 2\nu$, these models have block-diagonal genus-one partition functions, with the $\nu + 2$ representations of their *extended* algebra labelled by an integer l between 0 and ν (ν twice degenerate). Their fusion algebra is known to be described by the rules

$$\Phi_0 \cdot \Phi_l = \Phi_l \tag{4.1a}$$

$$\Phi_1 \cdot \Phi_l = \Phi_{l-1} + \Phi_l + \Phi_{l+1} \qquad 1 \leqslant l \leqslant \nu - 2 \tag{4.1b}$$

$$\Phi_1 \cdot \Phi_{\nu-1} = \Phi_{\nu-2} + \Phi_{\nu-1} + \Phi_\nu^{(+)} + \Phi_\nu^{(-)} \tag{4.1c}$$

$$\Phi_1 \cdot \Phi_\nu^{(\pm)} = \Phi_{\nu-1} + \Phi_\nu^{(\mp)} \tag{4.1d}$$

$$\Phi_\nu^{(+)} \cdot \Phi_\nu^{(-)} = \Phi_{\nu-1} + \Phi_{\nu-3} + \cdots \tag{4.1e}$$

with the other fusion rules deduced from those by the associativity property. In particular, it follows from (4.1b, d) that

$$\Phi_\nu^{(+)} + \Phi_\nu^{(-)} = (\Phi_1 - \Phi_0) \cdot \Phi_{\nu-1} - \Phi_{\nu-2} \tag{4.2a}$$

$$(\Phi_1 - \Phi_0).(\Phi_\nu^{(+)} + \Phi_\nu^{(-)}) = 2\Phi_{\nu-1}. \tag{4.2b}$$

From the orbifold picture, one expects this case to be related to the A_{2n+1} discussed in section 2, and the even Chebishev polynomials to be the appropriate objects. We set

$$\mathcal{P}_l(x) = P_{2l}(y) \qquad \text{where } x = y^2 - 1 \tag{4.3}$$

in such a way that

$$\mathcal{P}_0 = 1 \qquad \mathcal{P}_1(x) = x. \tag{4.4}$$

They satisfy the rules (cf the tensor multiplication by spin 1 in SU(2))

$$\mathcal{P}_1 \mathcal{P}_l = \mathcal{P}_{l-1} + \mathcal{P}_l + \mathcal{P}_{l+1}. \tag{4.5}$$

To complete the identification with (4.1), we have to define two polynomials $\mathcal{P}_\nu^{(\pm)}$ satisfying

$$\mathcal{P}_\nu^{(+)} + \mathcal{P}_\nu^{(-)} = \mathcal{P}_\nu \tag{4.1d'}$$

$$\mathcal{P}_\nu^{(+)} \cdot \mathcal{P}_\nu^{(-)} = \mathcal{P}_{\nu-1} + \mathcal{P}_{\nu-3} + \cdots \equiv \mathcal{Q}_\nu \tag{4.1e'}$$

and to impose on x the constraint that follows from (4.2)

$$\begin{aligned} \mathcal{R}_\nu &\equiv (x-1)\mathcal{P}_\nu - 2\mathcal{P}_{\nu-1} \\ &= \mathcal{P}_{\nu+1} - \mathcal{P}_{\nu-1} = 0. \end{aligned} \tag{4.6}$$

We have just seen that $\mathcal{P}_\nu^{(\pm)}$ have to be solutions of the equation

$$X^2 - X\mathcal{P}_\nu + \mathcal{Q}_\nu = 0 \tag{4.7}$$

and we shall indeed now prove that this equation admits polynomial solutions. Using (4.5), it is easy to see that one may factor out $(1+x)$ in the polynomial $\mathcal{R}_\nu$

$$\mathcal{R}_\nu = (1+x)Z_\nu$$
$$Z_\nu = \mathcal{P}_\nu - 2(\mathcal{P}_{\nu-1} - \mathcal{P}_{\nu-2} + \cdots + (-1)^{\nu-1}\mathcal{P}_0) \tag{4.8}$$

thus that $x^p Z_\nu = (-1)^p Z_\nu \bmod \mathcal{R}_\nu$ and that

$$Z_\nu(x)^2 = Z_\nu(-1)Z_\nu(x) \qquad \bmod \mathcal{R}_\nu. \tag{4.9}$$

Now, $\mathcal{P}_l(-1) = (-1)^l$ is established recursively, whence $Z_\nu(-1) = (-1)^\nu(2\nu+1)$ from which we conclude that

$$Z_\nu^2 = (-1)^\nu(2\nu+1)Z_\nu. \tag{4.10}$$

On the other hand, since the $\mathcal{P}$s are Chebishev polynomials (see (4.3)), one has

$$\mathcal{P}_l^2 = \mathcal{P}_0 + \mathcal{P}_1 + \cdots + \mathcal{P}_{2l} \tag{4.11}$$

('addition of two spins l'), but from (4.6) follows that $\mathcal{P}_{\nu+i} = \mathcal{P}_{\nu-i}$, $i = 1, \ldots, \nu$, whence

$$\mathcal{P}_\nu^2 = \mathcal{P}_\nu + 2(\mathcal{P}_{\nu-1} + \mathcal{P}_{\nu-2} + \cdots + \mathcal{P}_0). \tag{4.12}$$

We may thus assert from (4.10) and (4.12) that the discriminant of the equation (4.7) is a perfect square

$$\begin{aligned}\Delta &= \mathcal{P}_\nu^2 - 4(\mathcal{P}_{\nu-1} + \mathcal{P}_{\nu-3} + \cdots) \\ &= \mathcal{P}_\nu - 2(\mathcal{P}_{\nu-1} - \mathcal{P}_{\nu-2} + \cdots) \\ &= Z_\nu \\ &= \frac{(-1)^\nu}{2\nu+1} Z_\nu^2\end{aligned} \tag{4.13}$$

and that the two desired polynomials $\mathcal{P}_\nu^{(\pm)}$ read

$$\mathcal{P}_\nu^{(\pm)} = \tfrac{1}{2}\left[\mathcal{P}_\nu \pm \frac{\mathrm{i}^\nu}{\sqrt{2\nu+1}} Z_\nu\right]. \tag{4.14}$$

We conclude that the polynomials

$$\mathcal{P}_l \qquad 0 \leqslant l \leqslant \nu - 1 \qquad \mathcal{P}_\nu^{(\pm)}$$

form a representation of the fusion algebra of the $D_{2\nu+2}$ models of SU(2) current algebra. That the diagonalization of the $D_{2\nu+2}$ fusion rules involves imaginary coefficients whenever ν is odd is a well known fact (see for instance [14]). The last point that we want to make is that these polynomials are not independent on $\mathbb{Q}$ when $2\nu+1$ is a perfect square. In that case the representation of the fusion algebra by the $\mathcal{P}$s is not faithful. The first instance occurs for D_{10}, for which the combination of fields

$$X = 4\Phi_4^{(-)} - 2\Phi_4^{(+)} - 2(\Phi_3 - \Phi_2 + \Phi_1 - \Phi_0)$$

that has the property that for any field Y, $X \cdot Y = \lambda_Y X$, $(\lambda_Y \in \mathbb{Z})$, is represented by 0.

For illustration we provide explicit formulae for the D_4 and D_6 cases; the case of D_4 is a bit exceptional, as the generator of the ring is one of the 'degenerate' fields

$$D_4: \qquad \Phi_0 = 1 \qquad \Phi_1^{(+)} = X \qquad \Phi_1^{(-)} = X^2 \qquad X^3 = 1 \tag{4.15}$$

whereas D_6 exhibits the phenomena discussed above

$$D_6: \qquad \Phi_0 = 1 \qquad \Phi_1 = \phi \qquad \Phi_2^{(\pm)} = \tfrac{1}{2}\left[(\phi^2 - \phi - 1) \pm \frac{1}{\sqrt{5}}(\phi^2 - 3\phi + 1)\right]$$
$$\text{with } \phi^3 - 2\phi^2 - 2\phi + 1 = 0. \tag{4.16}$$

The cases of the E_6 and E_8 SU(2) theories are readily dealt with if one observes that their fusion algebras are, respectively, isomorphic to those of the $(3, 4)$ (Ising) and $(2, 5)$ (Lee–Yang) minimal models for which the discussion of the previous section applies.

5. Conclusion and questions

In this note we have shown that in several classes of rational conformal field theories the fusion ring may be represented as a ring of polynomials in one variable, quotiented by a certain polynomial constraint that may be integrated to a 'potential'. This extends to cases like the SU(3) models or the 'D' series of SU(2) models which were generally believed to require two-variable polynomials. In most cases, the solution to the Gepner conjecture does not appear to be unique, and even the degree of the polynomial constraint depends on the generator of the ring. In fact, this non-uniqueness shatters the naïve idea that was one of the original motivations of the present work, namely that this fusion potential might serve to *characterize* the fusion ring.

In minimal models, in particular, we have seen that there are quite a number of alternative ways of constructing this polynomial representation and the corresponding potential. Beside the information they encode on the fusion algebra, it would be quite interesting to find a *physical* (Landau–Ginsburg?) interpretation of these potentials.

Not all RCFTs admit such a representation of their fusion ring: for example the D_{10} orbifold whose representation is not faithful, or $SU(4)_2$ etc. It would be nice to have a characterization of those RCFTs that have this property.

In a forthcoming paper, we shall comment on some connections between this fusion potentials and the potentials that emerge from perturbed topological field theories.

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Appendix A

Given the rational numbers α_1 and α_2, let us look for rational solutions β_1 and β_2 to

$$e^{i\alpha_1} + e^{i\alpha_2} + e^{-i(\alpha_1+\alpha_2)} = e^{i\beta_1} + e^{i\beta_2} + e^{-i(\beta_1+\beta_2)}. \tag{A.1}$$

Multiplying both sides of (A.1) by $e^{i(\alpha_1+\alpha_2+\beta_1+\beta_2)/2}$, and taking the real and imaginary parts, we are left with the system

$$\sin\tfrac{1}{2}(\alpha_1-\beta_1)\sin(\alpha_1+\beta_1+\tfrac{1}{2}(\alpha_2+\beta_2)) + \sin\tfrac{1}{2}(\alpha_2-\beta_2)\sin(\alpha_2+\beta_2+\tfrac{1}{2}(\alpha_1+\beta_1)) = 0$$

$$\sin\tfrac{1}{2}(\alpha_1-\beta_1)\sin\tfrac{1}{2}(\alpha_1+\alpha_2+\beta_1)\sin\tfrac{1}{2}(\alpha_1+\beta_1+\beta_2) + \sin\tfrac{1}{2}(\alpha_2-\beta_2)\sin\tfrac{1}{2}(\alpha_1+\alpha_2+\beta_2)\sin\tfrac{1}{2}(\beta_1+\beta_2+\alpha_2) = 0. \tag{A.2}$$

Introduce the variables

$$\begin{aligned} a &= \tfrac{1}{2}(\alpha_1+\alpha_2+\beta_1) \qquad & b &= \tfrac{1}{2}(\beta_1+\beta_2+\alpha_1) \\ c &= \tfrac{1}{2}(\alpha_1+\alpha_2+\beta_2) \qquad & d &= \tfrac{1}{2}(\beta_1+\beta_2+\alpha_2) \end{aligned} \tag{A.3}$$

then equations (A.2) read

$$\sin(c-d)\sin(a+b) = \sin(b-a)\sin(c+d) \tag{A.4a}$$

$$\sin(c-d)\sin a\sin b = \sin(b-a)\sin c\sin d. \tag{A.4b}$$

Note that the vanishing of any sine factor on each side of (A.4*b*) (e.g. $c-d=b-a=0 \bmod \pi$) just amounts to taking one of the six trivial solutions of (A.1), namely (all the equalities are understood modulo π) $\beta_1=\alpha_1$, $\beta_2=\alpha_2$, or $\beta_1=\alpha_1$, $\beta_2=-(\alpha_1+\alpha_2)$ or the four permutations obtained by letting $\alpha_1 \leftrightarrow \alpha_2$ or $\alpha_1 \leftrightarrow -(\alpha_1+\alpha_2)$. Let us exclude these trivial solutions, then we can divide (A.4*a*) by (A.4*b*) to get

$$\frac{\sin(a+b)}{\sin a\sin b} = \frac{\sin(c+d)}{\sin c\sin d} \tag{A.5}$$

and rewrite (A.4*b*) as

$$\frac{\sin(b-a)}{\sin a\sin b} = \frac{\sin(c-d)}{\sin c\sin d} \tag{A.6}$$

hence we get $\cot^{-1} a = \cot^{-1} d$ and $\cot^{-1} b = \cot^{-1} c$, already excluded above. In conclusion the only solutions to (A.1) satisfy $\{\beta_1, \beta_2, -(\beta_1+\beta_2)\} = \{\alpha_1, \alpha_2, -(\alpha_1+\alpha_2)\}$ and reduce to $\beta_i = \alpha_i$ when the angles are constrained by (2.12). This completes the proof that the $SU(3)_k$ fusion rules may be represented by polynomials in one variable.

Appendix B†

For two given coprime integers p and q, and r and s such that $1 \leqslant r < p/2$, $1 \leqslant s < q/2$, we look for solutions r' and s' to

$$\cos\pi(r/p)\cos\pi(s/q) = \cos\pi(r'/p)\cos\pi(s'/q). \tag{B.1}$$

This amounts to

$$\frac{\cos\pi(r/p)}{\cos\pi(r'/p)} = \frac{\cos\pi(s'/q)}{\cos\pi(s/rq)} \tag{B.2}$$

the value of (B.2) belongs to the intersection of the extensions $\mathbb{Q}(\cos\pi/p) \cap \mathbb{Q}(\cos\pi/q)$, and is therefore rational. Let us show that indeed $\mathbb{Q}(e^{i\pi/p}) \cap \mathbb{Q}(e^{i\pi/q}) = \mathbb{Q}$. Set $\xi = e^{i\pi/pq}$, $\alpha = \xi^q$, $\beta = \xi^p$. Suppose we have

$$\sum_{i=0}^{2p-1} a_i\alpha^i - \sum_{j=0}^{2q-1} b_j\beta^j = f(\xi) = 0 \tag{B.3}$$

where a_i and b_j are rational. The equation $f(\xi) = 0$ is algebraic with rational coefficients, and is therefore satisfied by any conjugate ξ^c, $1 \leqslant c < 2pq$, c coprime with $2pq$. In particular let us choose those which preserve $\alpha = \xi^q$, namely take c to be of the form $c_k = 1 + 2kp$, k defined modulo q. It is straightforward to see that there exists a subset K of $\{0, 1, \ldots, q-1\}$, such that $\{c_k,\ k \in K\}$ runs over all the integers coprime with $2q$ modulo $2q$, which means that β^{c_k} runs over all the conjugates of β. Thus if we write

$$0 = \frac{1}{|K|}\sum_{k\in K} f(\xi^{c_k}) = \sum_{i=0}^{2p-1} a_i\alpha^i - \frac{1}{|K|}\sum_{k\in K}\sum_{j=0}^{2q-1} b_j\beta^{jc_k} \tag{B.4}$$

the second sum on the right-hand side of (B.4) is a symmetric function of β and its conjugates, therefore rational. Hence both sums in (B.3) are rational and we proved the statement.

We are now left with the problem of finding solutions to

$$\frac{\cos\pi r/p}{\cos\pi r'/p} \in \mathbb{Q} \tag{B.5}$$

where say $p = 2l + 1$ is an odd integer (otherwise exchange the roles of $p \leftrightarrow q$, $r \leftrightarrow s$ and $r' \leftrightarrow s'$). Consider the polynomial

$$\Pi(x) = \prod_{r,r'=1}^{p-1}\left(2x\cos\pi\frac{r'}{p} - 2\cos\pi\frac{r}{p}\right). \tag{B.6}$$

This polynomial has integer coefficients (as polynomial symmetric functions of $\{e^{i\pi a/p}$, $a = \pm 1, \ldots, \pm(p-1)\}$). Moreover it is monic, due to the well known identity

$$\prod_{r=1}^{p-1} 2\cos\pi\frac{r}{p} = (-1)^l \tag{B.7}$$

† This proof is entirely due to Michel Bauer

for odd $p = 2l + 1$, of degree $(p-1)^2$ and reciprocal, i.e. $\Pi(x) = x^{(p-1)^2}\Pi(1/x)$. The root of a monic polynomial with integer coefficients is what is called an algebraic integer; if it is rational, it has to be an integer (otherwise one could write it a/b, and one would have an equation $(a/b)^n = \text{integer}/b^{n-1}$, which is impossible). But if a is a root of $\Pi(x)$, then $1/a$ is also a root, therefore the only rational roots of $\Pi(x)$ are ± 1 and thus the only ways of realizing (B.5) are by taking $\cos \pi r/p = \pm \cos \pi r'/p$, i.e. $r' = r$ or $r' = p - r$, from which we deduce that $s' = s$ or $s' = q - s$. This completes the proof that the only possibly degenerate eigenvalue for $N_{(2,2)}$ in the (p, q) minimal conformal theory is zero.

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